ICLEHI 2015-6 Low Chee Soon

A Steep Battle between Flexible Knowledge and For-The-Test Mentality

Low Chee Soon
Centre for Pre-University Studies,
KDU College Penang
cslow@kdupg.edu.my

ABSTRACT

Literature and empirical studies abound, pointing to the potential benefits of teaching and learning flexible knowledge and strategy flexibility in mathematics education. Ironically, other studies on classroom practice frequently revealed little emphasis on flexible knowledge as a means to promote flexible connections with mathematical concepts for effective and efficient problem solving. This study administered seven multiple-solution tasks to thirty-two newly-enrolled A-Level student participants from twenty public secondary schools. The study aimed to assess the participants' flexibility in producing conceptually-varied solutions to the same tasks and adaptivity in employing relatively more efficient strategies. The participants' written solution strategies and group interview protocols served the main source of data for analysis. Findings generally pointed to the participants' low levels of flexibility and adaptivity. The ensuing interview identified for-the-test culture in the participants' past learning as the potential hindrance to the learning of flexible knowledge. While student performance is generally believed to be highly associated with learning experience, the findings have implied little emphasis on flexible knowledge in the participants' secondary studies. Further implications are discussed.

Keywords: flexible knowledge, for-the-test mentality, multiple solutions, strategy flexibility

Introduction

One interesting, lively aspect of learning mathematics is the availability of abundant resources and concepts which could be integrated to solve problems flexibly and adaptively. Numerous sources of literature and empirical studies have pointed to the potential benefits of teaching and learning flexible knowledge and strategy flexibility in mathematics education. In particular, encouraging exploration and comparison of alternative solutions would essentially increase the dynamism in mathematical thinking and be instrumental in promoting conceptual understanding, flexible knowledge and divergent thinking (Hopkins, 2010; Levav-Waynberg & Leikin, 2012; NCTM, 2000; Newton, Star, & Lynch, 2010; Rittle-Johnson & Star, 2007, 2009; Rittle-Johnson, Star & Durkin, 2009; Schoenfeld, 1985).

In fact, the values of inculcating flexible knowledge have long been recognized. According to Rogers (1983), learners should be given the freedom to learn and the autonomy to explore and discover knowledge. Educational effort should lead toward the abstention from the misleading image of 'problem solving' as absolute (a fixed way), complete (the only way) and permanent (an unchangeable way), which typifies a closure mentality—a barrier to flexibility and creativity. Fremont (1969) apparently holds an identical view and contends that learners should be shown varieties of solutions and given the freedom to choose methods of solutions, condemning teaching which restricts learners' freedom of solution choice.

Butts (1973) elaborates that the learning of arithmetic specifically and mathematics generally does not consist of merely memorizing a few hundred (seemingly independent) rules with the ultimate hope (seldom achieved) that one knows the right time to apply the right rule. Rather, the study of arithmetic should comprise the knowledge of a few basic principles and the development of ability to acquire skills and to solve problems by combining these principles in many different ways and in many different contexts—a salient emphasis on flexible knowledge. Expressing the same view as Butts, Kyle & Kahn (2009) contends that alternative approaches should be valued and rooted in the understanding of mathematics and its application. Peterson (1988) argues that instructional focus should not be merely on the answer, but the mathematics strategy for effective learning because "most higher-level mathematics responses have more than one right strategy that can be used to obtain the answer" (p. 11). On the same ground, Romberg, Zarinnia & Collis (1990) cited in Romberg (1993) base the importance of flexible knowledge on higher order thinking, which was characterized as being nonalgorithmic and complex and tending to yield multiple solutions, among others.

Considering alternative possibilities is a general heuristic found in psychological research on thinking and reasoning (Baron, 1988). There is a need to adopt a divergent approach which serves as a challenge for learners to look for a variety of ways to solve a problem, or a variety of consequences to a given problem. According to Dwyer & Elligett (1970), a divergent approach searches for situations in which diverse approaches are appropriate. And it is a crucial experience in the undergraduate education to have greater freedom for intellectual risk-taking and "playing with ideas", and for making a commitment to one of several possible solutions (Freeman, Butcher & Christie, 1971).

Unfortunately, such precious opportunity for learning flexible knowledge does not seem to have been much valued (Bingolbali, 2011). As a result, students rarely attempt alternative solutions to mathematical tasks (Stigler & Hiebert, 1999). Despite the substantial empirical evidences pointing to the positive potential of teaching and learning flexible knowledge (Alibali, 1999; Siegler, 1995; Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005; Levav-Waynberg & Leikin, 2012), the cultivation of mathematical literacy and conceptual understanding by explicit requests for alternative solutions to mathematical tasks is rare and uncommon. Schools appear to focus on those activities which would presumably guarantee better examination results (Lim, Fatimah, & Tan, 2004), with a flat-out aim to target at examination-style answering techniques (Popham, 2001). It seems that exploring alternative solutions to mathematical tasks has become a luxury with little significance for examination typically requires only single solutions. Unfortunately, teaching and learning single solutions to a task, while it is amenable to alternative strategies, could be greatly detrimental. Locked in a particular approach, students would lack the flexibility to adapt to new circumstances (Schoenfeld, 1987).

Flexibility, which is a key dimension of creativity, primarily refers to switching smoothly between different strategies (Guilford, 1959; Stein, 1974; Torrance, 1969). Some researchers however also included the notion of strategy efficiency in the definition of flexibility. For instance, Rittle-Johnson & Star (2007), Star & Rittle-Johnson (2007), and Star & Seifert (2006) have defined flexibility in problem solving as the knowledge of both multiple strategies and the relative efficiency of the strategies. In their studies, they assessed students' solution steps in solving linear equations. The numbers of steps involved with various solution strategies, such as "expand first" or "divide first" in solving a linear equation, e.g. $3(x + 5) = 21$, enable the comparison of strategy efficiency, which is viewed as an intrinsic quality of flexibility. However, strategy efficiency obviously may serve only one of the many adaptive qualities of mathematical solutions. An adaptive choice of solution certainly relies on various factors such as subjectivity and contexts. Verschaffel, Luwel, Torbeyns, & Dooren (2009), based on an extensive literature review, operationally define flexibility as use of multiple strategies, and adaptivity as selection of most appropriate strategy. The term 'appropriate', however, has not been specifically defined, but is broadly referred to as (a solution choice) dependent on the task in hand, for that particular problem solver and in a particular context (Verschaffel et al., 2009).

In this study, flexibility is perceived as the ability to produce alternative solutions to a task, while adaptivity the ability to employ relatively more efficient solutions. Furthermore, it is posited that as the level of complexity increases, gauging a solution by means of algebraic flexibility or number of steps per se does not seem to be adequate and appropriate. Solutions of higher-level (i.e. upper secondary level and A-Level) mathematics may involve substantial mathematical concepts which could be flexibly integrated and applied. As such, this study founded flexibility on the use of conceptually-varied solution strategies. Similarly, adaptivity was referred to as conceptual efficiency, which indicates the relative efficiency of a solution based on the amount of mathematical concepts involved in the solution. Specifically, this study set out to examine A-Level students' mathematical flexibility and adaptivity and to identify the reasons for the observed outcomes. Of particular interest was the students' ability to integrate their previously-learned mathematical concepts in producing multiple solutions to the same tasks, in the domains of Quadratics and Geometry. The study was guided by the questions: To what extents would the participants exhibit strategy flexibility (i.e. the ability to produce conceptually-varied solutions to the same tasks) and adaptivity (i.e. the ability to produce conceptually efficient solutions)? What are the reasons underlying the observed performance?

Method

Research design

This study adopted a case study qualitative research design, whereby A-Level participants' solution strategies employed for seven multiple-solution tasks were analyzed. In particular, the levels of flexibility and adaptivity exhibited in their solution strategies were determined. The participants were later streamed on their levels of flexibility (i.e. low, medium and high levels). Possible reasons for their performance were then probed via three semi-structured group interviews. Group interviews were considered in view of cost-effectiveness and efficiency in addition to possible collection of shared understanding and sophisticated data arising from the interactions among the participants (Gay, Mills, & Airasian, 2009).

Participants

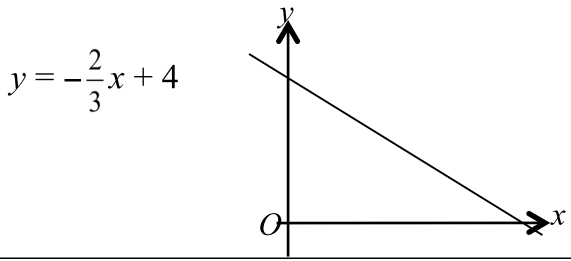
This study involved thirty-two A-Level students (16 males and 16 females) who newly joined a private college upon completing their secondary studies at twenty public secondary schools. Majority of the participants were high achievers in Mathematics according to their Sijil Pelajaran Malaysia (SPM) mathematics results. As such, it was hypothesized that the participants would have little difficulty in producing accurate answers to the administered tasks. It would however be uncertain if they could integrate and apply their previously-learned mathematical concepts to solve the administered tasks flexibly and adaptively. It was then interesting to observe if the participants' flexibility and adaptivity in mathematical problem solving would be a concomitant of their SPM mathematics results.

Measure

Seven school-based multiple-solution tasks (Table 1) were designed and used in this study as the key instrument for data collection. The tasks were aimed at investigating if the participants were able to produce alternative solutions and employ relatively more efficient strategies in tackling the tasks. The seven mathematical tasks, which primarily focused on Quadratics and Geometry, were designed based on three main criteria: (i) the tasks are multiple-solution tasks which could be solved in two or more ways; (ii) the tasks are generally solvable with secondary-level mathematical concepts (see SPM Mathematics syllabuses at http://www.moe.gov.my); and (iii) the tasks would not be overly demanding as to stifle the participants' effort in attaining accurate answers. The third criterion was deemed to be necessary so that failures resulting from the overwhelming effect of task complexity could be minimized, if not completely avoided, making it possible to attribute failures to inflexibility without the interference of task complexity. Quadratics is not as simplistic as linear equations, nor is it as complex as other nonlinear, transcendental functions. It was hence considered as an ideal domain for achieving the research objectives. Similarly, geometrical tasks are generally subjected to multiple interpretations and representations, thus are amenable to various strategies and more likely to elicit flexible knowledge.

Table 1 Seven multiple-solution tasks for assessing the A-Level participants' flexibility and adaptivity

1	Solve the inequality: $(2x - 5)^2 - 1 > 0$.
2	Solve the equation: $x - 2\sqrt{x} - 15 = 0$.
3	Given that the line, $y = k$, where k is a constant, shares only a common point with the curve, $y = 5 + 4x - x^2$. Find the value of k.
4	The position vectors of points P and Q are $2\mathbf{i} + 5\mathbf{j}$ and $-8\mathbf{i} + 3\mathbf{j}$, respectively. The point R divides the line segment PQ internally in the ratio 3 : 1. Find the coordinates of R.
5	is a parallelogram, whose three vertices have coordinates $P(2, 5)$, $Q(6, 3)$ and $R(10, 13)$. Determine the coordinates of the point S.
6	Find the acute angle between the lines $y = 2x + 5$ and $y = 5x + 2$.
7	The diagram shows the line of equation $y = -\sqrt{x}x + 4$. Find the shortest distance from the origin O to the line. Express your final answer to three significant figures.



Coding of multiple-solution strategies with clusters of mathematical concepts

Based on Leikin's (2007) conception of solution spaces, it was considered in this study that there exist multiple solutions to a task which make up a space (set) of alternative strategies for solving the task. Alternative strategies for a task may entail different clusters of mathematical concepts integrated and applied collectively to give rise to conceptually-varied solutions. In other words, a *solution strategy* for a task refers to the cluster of mathematical concepts employed in solving the task. The *conceptual efficiency* of a solution strategy is concerned with the relative amount of mathematical concepts involved in implementing the solution strategy. That is, a solution strategy of higher conceptual efficiency involves relatively fewer mathematical concepts and could be implemented more easily.

Prior to data collection, possible solution strategies for the multiple-solution tasks in Table 1 were generated and coded based on the clusters of mathematical concepts involved. As such, two solution strategies for solving a task are identified as different if they involve different clusters of mathematical concepts. For instance, to solve the inequality $(2x - 5)^2 - 1 > 0$, a problem solver may first expand $(2x - 5)^2$, followed by simplification to and factorization of $x^2 - 5x + 6$, before performing a functional sign analysis by graphical means. On the other hand, the inequality could also be quickly solved by the reasoning that $(2x - 5)^2 \geq 0$, for any real values of x. Thus, the magnitudes of $(2x - 5)$ must be sufficiently large to be greater than 1, hence the inference '$2x - 5 < -1$ or $2x - 5 > 1$'. Apparently, the first strategy requires relatively more mathematical concepts to reach the goal state. It is hence less conceptually efficient compared with the second strategy.

The conceptual validity and accuracy of all the identified solution strategies for the tasks were independently validated by two content experts, who were newly-retired teachers and possessed at least 20 years of experience in teaching SPM Additional Mathematics. The experts confirmed the tasks could be solved in two or more ways with concepts within the SPM Mathematics syllabuses. The experts were asked to suggest new solution strategies for the tasks, if available, and arrange all the known solutions to each task in ascending order of conceptual efficiency based on the relative amounts of mathematical concepts involved.

Consequently, the relative conceptual efficiency of each possible pair of solution strategies was determined by sequential comparison, aimed at meeting the expectations of at least two of the researcher and the content experts. The validation process finally gave rise to some five to twelve solution strategies for each of the seven tasks, all arranged in ascending order of conceptual efficiency. The validated solution strategies served as a basis against which the levels of flexibility and adaptivity of the participants' solutions were determined.

Procedure

The multiple-solution tasks in Table 1 were administered to the thirty-two A-Level participants before the teaching of the A-Level course was started. Initially, the test requirements were briefed and utmost confidentiality ensured. It was clarified that different solutions to the same task may involve different mathematical concepts and solution strategies. The participants were required to attempt the mathematical tasks in as many ways as possible and spend no more than 3 hours in attempting all the seven tasks administered. The 3-hour duration was deemed to be sufficient based on the study by Low & Chew (2013), in which the durations of attempt at eight multiple-solution tasks were recorded to range from approximately 50 minutes to 1 hour 42 minutes. In this study, the durations ranged from approximately 1 hour 49 minutes to 2 hours 44 minutes, with majority of the participants having completed the test in 2 hours 15 minutes. The entire test session was videotaped.

Table 2

The Scoring Rubric for Assigning Flexibility Score to a Solution Strategy

Descriptor	Flexibility Score
Non-attempt, a repeated or trivial solution, conceptually invalid solution	0
Reasonable mathematical attempt initiated with premature termination, showing no clear indication of any valid strategy	1
A well-structured solution strategy with incorrect answer owing to invalid assumption, i.e. assuming perpendicular diagonals for a parallelogram	2
A valid, well-structured solution strategy with faulty problem representation or interpretation, i.e. improper labeling of the vertices of a quadrilateral	3
A valid, well-structured solution strategy with only minor misconceptions, i.e. non-verification of solution; or with pure computational or transcription errors	4
A valid, well-structured solution strategy accurately presented	5

Upon analyzing the written responses to the seven mathematical tasks, the thirty-two participants were distributed into three different groups, based on their stratified levels of flexibility. Three follow-up, semi-structured group interviews were then conducted a week after the test administration. All three group interviews were both audio- and video-recorded and transcribed for further analyses.

Coding and data analysis of participants' solution strategies and group interviews

During the coding process, each solution strategy to a task was assigned a flexibility score (0 to 5) based on its accuracy and quality as depicted in Table 2. Flexibility scores 0 to 2 generally point to invalid solutions, while scores 3 to 5 refer to valid solutions of varying qualities in terms of accuracy. Subsequently, the level of flexibility (0 to 5) for each task was determined based on both the number of valid solutions produced and the number of solutions with a flexibility score of at least 4, as shown in Table 3. As such, a higher level of flexibility would imply the ability to produce more conceptually-varied solution strategies with higher accuracy. The study by Low & Chew (2013) and this current study both revealed that participants hardly produced four or more conceptually-varied solution strategies, thus the ability to generate four or more solution strategies for a task with flexibility scores of at least 4 was considered as a relatively high level of flexibility (Table 3).

Table 3

The Rating of Flexibility Level on each Task

Joint Measure		
No. of Valid Solution Strategies	No. of Flexibility Scores ≥ 4	Level of Flexibility
0 – 1	0	0
1	1	1
≥ 2	0	1
	1	2
	2	3
	3	4
	≥ 4	5

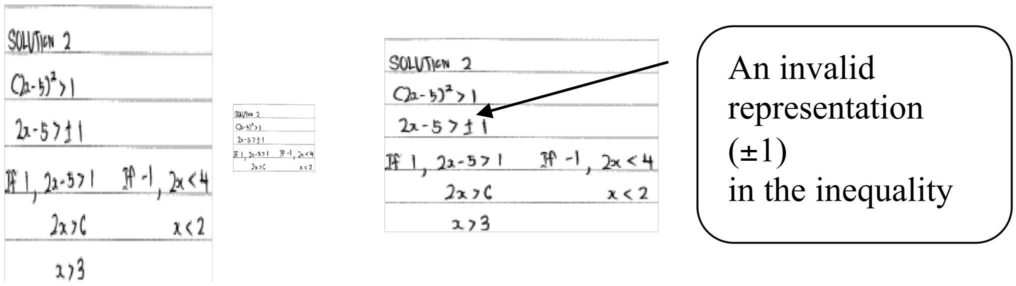


Figure 1

A participant's solution strategies for Task 1: Solve the inequality $(2x-5)^2-1>0$

In addition, the level of adaptivity was assigned for each task based on the highest possible conceptual efficiency of the solutions to the task measured relative to the pre-determined solution strategies for the task arranged in ascending order of conceptual efficiency. For instance, there exist five solution strategies, hence five possible levels of adaptivity for Task 1: 'Solve the inequality: $(2x-5)^2-1>0$'. For illustration, a participant's solution strategies for Task 1 are shown in Figure 1. The Solution 1 is perfect and accurate, while the Solution 2 contains a representational error in the inequality '$2x-5>\pm1$'. They were hence assigned flexibility scores of 5 and 4 respectively, giving a flexibility level of 3 (i.e. two valid solutions with flexibility scores of at least 4). The level of adaptivity was coded as 5 because of the Solution 2 which was relatively the most conceptually-efficient solution measured against the validated strategies.

Upon streaming the participants into low, medium, and high levels of flexibility, three semi-structured group interviews were conducted. The interviews were both audio- and video-recorded and transcribed. The verbal transcripts of the three interviews were reviewed multiple times and coded for subsequent analyses (Creswell, 2014; Gay et al., 2009; Merriam, 1998). Related codes were then collapsed into themes guided by both the research questions and emerging issues deemed to be educationally significant and relevant.

Reliability assessment

For reliability assessment, the cases were arranged in ascending order of flexibility. Twelve scripts were then selected based on a stratified random sampling method and subjected to reliability tests. The sampled scripts were independently rated by the same two content experts mentioned earlier, in addition to the ratings by the researcher. Kappa measure of agreement, which takes into consideration the raters' agreements by chance, was computed across tasks using the SPSS 17 (Pallant, 2013). The Kappa measures of agreement for flexibility and adaptivity were found to be at least 0.913 and 0.922 (n = 84, p < 0.0005), among the three raters, indicating very good inter-rater agreement. In addition, the transcribed protocols from all the group interviews, including data interpretation, were subjected to member checking (Creswell, 2014).

Findings

Levels of flexibility and adaptivity

In general, the findings point to low levels of flexibility and adaptivity in the participants' use of solution strategies. Tables 4 and 5 respectively show the numbers of participants who attained the various levels of flexibility and adaptivity across tasks.

Table 4
The numbers of participants with varying levels of flexibility

Task	Level of Flexibility						Rate of successful attempt	
	0	1	2	3	4	5		
1	11	14	0	7	0	0	21	65.6 %
2	25	4	0	3	0	0	7	21.9 %
3	18	12	0	1	1	0	14	43.8 %
4	9	21	0	2	0	0	23	71.9 %
5	12	16	0	4	0	0	20	62.5 %
6	26	6	0	0	0	0	6	18.8 %
7	19	12	0	1	0	0	13	40.6 %

Table 5
The numbers of participants with varying levels of adaptivity

Task	Level of Adaptivity											
	Max	1	2	3	4	5	6	7	8	9	10	11	others
1	5	1	19	0	0	1							
2	6	0	0	4	0	0	2						1
3	6	1	8	0	2	0	3						
4	5	0	0	0	0	23							
5	6	1	0	5	1	2	11						
6	10	1	1	3	0	0	0	1	0	0	0		
7	11	0	0	0	0	1	6	0	0	4	2	0	

Majority of the participants attained level of flexibility 0 or 1 for most of the tasks, implying that they either could not solve the tasks or they could hardly produce more than one valid solution to a task. They could better solve more familiar tasks (i.e. Tasks 1, 4, and 5) but performed rather poorly with unfamiliar tasks (i.e. Tasks 2 and 6).

The participants attained low to moderate levels of adaptivity for most of the tasks with solution strategies found to be largely convergent. In addition, they were found to be particularly good at solving Task 4, simply with the use of formula. In general, signs of exploration and reasoning with effortful attempts to integrate learned concepts were hardly observed.

Reasons for the Participants' Low Performance

The transcribed group interviews were analyzed and organized into themes according to the research questions and emerging issues. Numerous themes have indeed emerged. This paper reports on the reasons for the participants' low levels of flexibility and adaptivity, which appeared to be prominently associated with examination requirements and for-the-test culture. The themes reported herein include: *past learning experience*, *exam-oriented mentality*, and *intense practice and low retention*. In the subsequent presentation, pseudonyms (in the form of initials) are used to protect the participants' identity. The letters L, M, and H for low, medium, and high, respectively indicate the relative levels of flexibility among the participants.

Past learning experience: The seemingly lack of exposure to flexible knowledge in the participants' past learning experience seemed to be a key reason for low flexibility and adaptivity. When asked why the participants mainly produced single solutions during the multiple-solution test while the tasks could be solved in two or more ways, Lv (L) explicitly shared that "... because in secondary school, we had been trained to just find a solution to each question instead of diverging our answers into several solutions. We brought it from secondary school." A similar argument was expressed by Kx (L), "I think we were not taught to do in more ways. We just focused on finding the answers and get it right. Then it's ok already." Kx further explained, "Our perspective is we just do the question, we settle it, then, just finish, no need to understand." Similarly, Bar (M) and Ck (H) both admitted that they mostly attempted only one way in solving a task because "that is the way I've been learning all the while. So I had no (other) ways already ..." as remarked by Bar. In the same tune, Ck lamented that "... most of the questions, I can only produce one solution. It's because I think I'm not really exposed to many many ways of solving it. In school, we were only taught of one strategy ...." All these remarks sounded as if doing mathematics is merely to apply a learned (mechanistic) procedure to produce the required answer. Such approach to learning mathematics with single solutions seemed to suggest a predominant emphasis on procedural instead of conceptual and conditional knowledge. Similarly, when the participants were asked if they ever had any habitual inclination toward or experience in attempting mathematical tasks in different ways, Hc (L) and Abi (H) respectively admitted, "Never" and "Rarely", without hesitation. Such situations could have been more damaging if the learning of mathematics was somehow narrowed to merely facts and "formulas, that we have to memorize." [Ew (H)] The lack of experience in dealing with multiple-solution tasks could have resulted in a low predilection for thinking divergently in problem solving. Interestingly, exploring multiple solutions seemed to be a luxury which would only be practiced conditionally, as implied by Bar (M), "I only tried different ways if I don't get the answer. If I get the answer, then I'll stop there already, because I've already got it." Such conditional approach to tackling multiple solutions seemed to be greatly result-oriented.

Apparently, the lack of opportunity in divergent exploration has contributed to low flexibility and adaptivity particularly for the participants from the L and M groups. On the other hand, some participants from the H group seemed to have been taught flexible knowledge in school. Their ability to reach out to multiple solutions, however, appeared to have been dampened by what Lim et al. (2004) term as "for-the-test-mentality", which will be further discussed under the following themes.

Exam-oriented mentality: Examinations and the requirements as stipulated in the marking schemes for high-stake assessments seemed to have a strong restrictive bearing on the participants' learning behavior and performance. Interestingly, the relevant concerns were mainly brought up by the H participants, who were probably more meticulous and detailed in handling their past examinations. Wil (H) commented, " ... in my secondary school, we need only one type of solution for, to manage our exams. So, practically after I've learned one, and I'm get used to it, I practically forget others." Apparently, the requirement for single solutions in examinations has directly or indirectly orientated learners towards focusing on single solutions, which suffice in tackling examinations. With agreement in gesture from other group participants, Wil elaborated further with an example, indicating the influence of examination requirement on how a solution should be structured in order to gain marks in the examination. Wil explained, "... we are not allowed to use short-cuts from the past five years … like completing the square, … one of the, you know, marking is, we are, we are to write out the minus b over 2 squared first [the 'first' was emphatically accented], that carries one mark. If you don't write it, then no marks for you … " Wil's arguments were supported by Abi (H), Ky (H) and Jac (H) and others in gesture. Abi added, "Because each line consists, may consist a mark, you see, especially for our second paper, Add. Maths, each line will have a mark, and if let's say you miss one line, you probably lose some marks. It actually costs a lot." Such rigidity could have inevitably restrained personal exploration!

Interestingly, the high-achieving students (i.e. the H participants) seemed to carry with them high self-efficacy. A few participants insisted that they actually had flexible knowledge. When asked if they were actually aware of multiple solutions, but decided to put down only single solutions during the multiple-solution test, Abi and Wil (both from H) replied without hesitation, "Yes." Some others nodded in agreement. But when probed further as to why they did not show multiple solutions which they claimed to have known, they just laughed. They then came up with all kinds of reasons, such as "Because we've got stuck half-way." [Ck (H)] and "… we remembered the first part, and then not the second part of the solutions … we've got stuck somewhere … and I just canceled off everything (leaving only single solutions)." [Wil (H)] Wil later explained further that even if he had learned multiple solutions from his previous teacher, he would apply only the solution he found easiest, constantly employed, and finally got used to and would forget about all other possible solutions. Wil explained, "We're already used to the most easier and convenient method to us, as for me lar. I mean this is the way that I'm most used to, and I always do this way, and then you suddenly, like you want me to explore other ways, of course I'll face difficulty, but I'll try it out and somehow I get blocked, and then I'll just stick with one way." Wil also creatively likened his "one way" to an "express highway" as an efficient means compared with other less efficient strategies. Wil's remarks were later supported by Ck (H) who asserted, "And we apply (only) whatever that we are more comfortable with only in the exams."

The examination requirement for only single solutions were likely to have thwarted the teacher's effort in cultivating divergent/flexible thinking, in addition to minimizing the student's propensity for analysis and reasoning during the problem-solving process. Such scenario is believed to have greatly undermined student's capacity to react fruitfully to solving problems, particularly in new and challenging situations. It seems highly likely that over-emphasis on examinations and results solely based on accuracy has unhealthily encouraged the building of converging knowledge base rather than diverging explorative behavior in the learning of mathematics.

Intense practice and low retention: While most mathematics experts are likely to argue that mathematical problem solving requires analysis and reasoning, and thus is not so much dependent on memorization, the participants appeared to have resorted to memorization in their past learning. As such, they seemed to be faced with poor retention over time. A few participants defensively explained that they could not produce (multiple) solutions to certain tasks because they could not remember or recall any possible ways to solve the tasks. For instance, Vos (L) lamented that "... behind a four months' break after schooling life, so we've totally forgot(ten) about it since we haven't practiced anything for four months." This was supported by Kr (L), "Totally forgot ... we forgot the methods, forgot how to do .... Got some basics of how to do, (but) we don't know how to expand it." The same was argued by Ck (H), "... we never touch maths for very long time. So it gets rusty, even the basics get rusty ...." And this was echoed by Abi (H), "... for more than three months we didn't touch Maths anything. So, (it's) kind of rusty up here, so we don't remember any of the formulas, and you just do what you can do, and most probably that is how you can solve questions by only one solution." Abi's remark also sounded as if doing mathematics is merely a matter of applying formulas.

While being probed in greater detail as to the consequence of having lost touch with mathematics over the past few months, Kr (L) indicated that it was not so much concerned with mathematical concepts and knowledge, but *application* which could be a main reason for low flexibility and adaptivity, as remarked by Kr, "Applying is the problem." Kr later elaborated, "Application questions ... most of us when we were studying for SPM, we do a lot of exercise, you know. So now after four or five months, we never do. Obviously we forget everything lar." When asked about any situation which could probably help remember concepts better and longer, Abi (H) replied with "Doing the questions again and again" which was supported by Wil (H), "Ya, practice makes perfect." Intense practice appeared to be a must-have and must-work strategy. When asked further about the need for understanding and reasoning in addition to regular practice, Abi responded, "Because we need the formula to be stuck in our heads. So, we need to do it again and again and again so that it can be stuck in our heads." No doubt, sufficient practice is necessary for good retention; however, there also appeared to be a severe lack of heuristic skills and well-integrated theoretical knowledge, conceptual understanding and the ability to apply learned concepts. Perhaps, low conceptual understanding and little experience in exploring and applying mathematical concepts in arriving at alternative solutions could have resulted in both low retention rate as well as incapacity to be flexible and adaptive in mathematical attempts. Moreover, it is bewildering as to how the participants' intense practice in preparation for their high-stake SPM national examinations could have led to a fast-fading memory lasting less than a few months. Apparently, intense practice (based on a behaviorist model), despite being effective in tackling examinations, is far from being sufficient as a long-term solution for the nurturance of mathematical thinking and sound problem-solving skills. Sadly, the participants hardly touched on the need for exploration, analysis and reasoning in solving mathematical tasks. It appeared that the participants tended to rely on prior knowledge of mathematical procedures, which became vague with time, instead of exploration, analysis and reasoning in dealing with the multiple-solution tasks. This phenomenon probably explains the low retention rate as a consequence, which in turn led to low flexibility and adaptivity in the multiple-solution test.

Discussion and Conclusion

This study investigated A-Level participants' strategy flexibility and adaptivity in their attempts at seven multiple-solution tasks which were generally solvable with (learned) secondary mathematics. Specifically, the study aimed to determine the extent to which their solutions exhibit flexibility and adaptivity (namely, the ability to employ alternative solution strategies for the same task and the ability to apply relatively more efficient strategies) and identify the reasons underlying the participants' performance. Flexibility and adaptivity are two critical attributes in problem solving (Baroody, 2003; NCTM, 2000). It is generally recognized that being flexible and adaptive implies the cognitive capability to transfer, connect and integrate relevant concepts during a problem-solving process.

The participants' solutions generally revealed low flexibility with single solutions to more familiar tasks and low performance with less familiar tasks. In general, the participants were unable to establish connections and integrate relevant (learned) concepts satisfactorily for solving multiple-solution tasks flexibly and adaptively. Few participants produced conceptually-varied and adaptively-efficient solutions. Findings generally point to students' inadequate relational skills to penetrate cross-topical (mathematical) domains for effective problem solving. It appeared that a predilection for problem analysis and reasoning was absent, being overshadowed by a behavioral approach to problem solving—strongly driven by *recalling* learned (packaged) solutions to specific tasks instead of *mobilizing* learned (isolated, but related) concepts to be integrated conditionally. As most solutions smacked of low flexibility, the findings suggest that even school high-achievers may have difficulty in connecting, integrating and applying *learned* mathematical concepts. Those solutions with relatively higher rates of successful attempt appeared to converge toward single, typical strategies. Schoenfeld (1992) remarks that it is the novice rather than the experienced, expert problem-solvers who are often associated with ineffective use of the mathematical knowledge they have available. From this perspective, it could be concluded that academically high achievers might not necessarily equal effective problem solvers. Nonetheless, the findings carry a low external validity in view of the small sample size. Thus the results could not be generalized even if low flexibility and adaptivity is believed (and likely) to be prevalent among students.

An over-emphasis on attaining right answers in examinations, commonly referred to as for-the-test mentality, appeared to be a key reason for the observed phenomena. With such mentality students are probably more likely to resort to intense practice and memorization in the learning of mathematics. Despite the various legitimate reasons for high-stake examinations, some undesirable effects of examinations seemed to have interfered with school learning through teaching to the test, in general, and mathematics learning, in particular (Popham, 2001; Natriello, 2009). One apparent impact seemed to be intense practice for the sole purpose of achieving enhanced familiarity and prompt solutions which could have dampened the predilection for exploring alternative solutions. The participants' converging use of more efficient strategies coupled with low flexibility is likely to suggest the prevalent influence of students' past learning experience which focused on single, adaptively efficient solutions, probably as a strategic means to excel in high-stake national examinations. Encouraging attempts at and discussion of alternative solutions in the learning of mathematics is an attribute of good teaching practice (Chambers, 2008). However, in traditional (and probably even current) practice, teachers usually determine the way students should answer a question in an examination, whereby the joy and complexity of problem solving are replaced with uninteresting, simple retrieval of learned facts. Apparently, learning for examinations and learning to understand mathematics as a flexible instrument in problem solving have been far from a balance.

The lack of emphasis on flexible knowledge is likely to have a strong bearing on students' mathematics learning and achievement. The impact of learning single solutions, worst still with a lack of proper reasoning, is highly malevolent. Students might believe that there is always one solution to a task (Schoenfeld, 1985), develop low transfer skills owing to little experience in mobilizing equally-robust concepts in arriving at various solutions, achieve low conceptual understanding not knowing the learned concepts could indeed be applied in many situations other than those illustrated in the same topic, and probably worst of all, establish a rigid bond between specific solutions and tasks of similar nature, resulting in inflexibility and difficulty in subsequent learning and performance (Alibali, 1999). Fueled by intense procedural practice, the students might develop a habit to solve a task instantaneously with a familiar solution, seemingly without the need for task analysis and reasoning. While such problem-solving behavior may still be effective for solving similar, familiar tasks, it is far from being sufficient in dealing with unfamiliar, novel problems. According to Ericsson, Prietula, & Cokely (2007), it is the deliberate, effortful practice on tasks *beyond one's level of competence and comfort* which actually leads to enhanced and superior capability and performance. School students usually practice on tasks of *similar* features with *known* solutions—a phenomenon Schoenfeld (1985) termed as exercise, *not* problem solving, owing to the absence of intellectual impasse. The effects would only be an enhanced familiarity with particular kinds of tasks with little or no gain in cognitive advancement. Students are led into thinking that mathematical problem solving is merely know-how of fixed ideas and algorithms, without the needs for heuristic skills, reasoning, interpretation, analysis, and planning.

Perhaps, it has reached the time to be bold to accept the extant rigidity in students' problem-solving capability which has been overshadowed by the increasingly sterling academic results. As remarked by Kilpatrick (1985), "one cannot expect to accomplish one goal in problem solving by teaching for another." (p. 11) Certainly, flexible and adaptive knowledge has to be an educational goal before students could become flexible and adaptive. However, teacher's choices of tasks in classroom delivery are to a large extent shaped by the examination requirements, which could well be instrumental for improving mathematics learning (Yeap, 2010). Further studies may look into the possibility of incorporating multiple-solution requirements in examinations and probably also on the impacts of students' increased experience in dealing with multiple-solution tasks which call for cross-topical domain knowledge.

References

Alibali, M. W. (1999). How children change their minds: Strategy change can be gradual or abrupt. *Developmental Psychology, 35*(1), 127–145.

Baroody, A. J. (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In A. J. Baroody and A. Dowker (Eds.), *The development of arithmetic concepts and skills: constructing adaptive expertise* (pp. 1–33). London: Lawrence Erlbaum Associates, Publishers.

Baron, J. (1988). Thinking and deciding. Cambridge: Cambridge University Press. Cited in Garnham, A. and Oakhill, J. (1994). *Thinking and reasoning.* UK: Blackwell Publishers.

Bingolbali, E. (2011). Multiple solutions to problems in mathematics teaching: Do teachers really value them? *Australian Journal of Teacher Education, 36*(1), 18–31.

Butts, T. (1973). *Problem solving in mathematics*. London: Scott, Foresman and company.

Chambers, P. (2008). *Teaching mathematics: Developing as a reflective secondary teacher.* London: SAGE Publications Inc.

Creswell, J. W. (2014), *Educational research: Planning, conducting and evaluating quantitative and qualitative research* (4th Ed.). England: Pearson Education Limited.

Dwyer, R. C., & Elligett, J. K. (1970). *Teaching children through natural mathematics*. New York: Parker Publishing Co., Inc.

Ericsson, K. A., Prietula, M. J., & Cokely, E. T. (2007). The making of an expert. *Harvard Business Review, 85*(7/8), 114–121.

Freeman, J., Butcher, H. J., & Christie, T. (1971). *Creativity: A selective review of research* (2nd Ed.). London: Society for Research into Higher Education Ltd.

Fremont, H. (1969). *How to teach mathematics in secondary school*. London: W. B. Saunders Company.

Gay, L. R., Mills, G. E., & Airasian, P. (2009). *Educational research: Competencies for analysis and applications* (9th Ed.). London: Pearson.

Guilford, J. P. (1959). The three faces of intellect. American Psychologist, 14, 469–479. Cited in Huffman, K., Vernoy, M. and Vernoy, J. (1994). *Psychology in action* (3rd ed.). New York: John Wiley & Sons, Inc.

Hopkins, K. R. (2010). *Teaching how to learn in a what-to-learn culture*. San Francisco: Jossey-Bass.

Kilpatrick, J. (1985). A retrospective account of the past 25 years of research on teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical solving: Multiple research perspectives* (pp. 1 – 15). London: Lawrence Erlbaum Associates, Publishers.

Kyle, J., & Kahn, P. (2009). Key aspects of teaching and learning in mathematics and statistics. In H. Fry, S. Ketteridge and S. Marshall (Eds.). *A handbook for teaching and learning in higher education: Enhancing academic practice* (3rd Ed.). New York: Routledge.

Leikin, R. (2007). Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks. In *The fifth conference of the European Society for Research in Mathematics Education – CERNE-5* (pp. 2330–2339).

Levav-Waynberg, A., & Leikin, R. (2012). The role of multiple solution tasks in developing knowledge and creativity in geometry. *The Journal of Mathematical Behavior, 31*(1), 73–90.

Lim, C. S., Fatimah, S., & Tan Sok Khim (2004). Culture of mathematics learning in Malaysia: Two case studies. In J. Wang and B. Xu (Eds.), *Trends and challenges in mathematics education* (pp. 139–157). Shanghai: East China Normal University Press.

Low, C. S. and Chew, C. M. (2013). Flexibility and Adaptivity in Mathematical Problem Solving: Some Observations. Presented at the 5thInternational Conference on Science and Mathematics Education: Empowering The Future Generation Through Science and Mathematics Education,RECSAM, 11–14 November, 2013.

Merriam, S. B. (1998). Qualitative Research and case study applications in education. San Francisco: Jossey-Bass.

Merriam, S. B. (1988). Case study research in education: A qualitative approach. San Francisco: Jossey-Bass.

National Council of Teachers of Mathematics (2000). Principles and standards for school mathematics. Reston, VA: NCTM.

Natriello, G. (2009). High stakes testing and teaching to the test. In L. J. Saha, & A. G.Dworkin (eds.), International Handbook of Research on Teachers and Teaching (pp. 1101–1111). Springer Science + Business Media LLC.

Newton, K. J., Star, J. R., & Lynch, K. (2010). Understanding the development of flexibility in struggling algebra students. *Mathematical Thinking and Learning, 12*, 282–305.

Pallant, J. (2013). *SPSS survival manual: A step by step guide to data analysis using IBM SPSS* (5th Ed.). England: McGraw-Hill Education.

Peterson, P. L. (1988). Teaching for higher-order thinking in mathematics: The challenge for the next decades. In D. A. Grouws, T. J. Cooney, & D. Jones (Eds.), *Perspectives on research on effective mathematics teaching* (pp. 2–26). The United States: National Council of Teachers of Mathematics, Lawrence Erlbaum Associates.

Popham, W. J. (2001). Teaching to the test? *Educational Leadership,58*(6), 16–20.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561–574.

Rittle-Johnson, B., & Star, J. R. (2009). Compared to what? The effects of different comparisons on conceptual knowledge and procedural flexibility for equation solving. *Journal of Educational Psychology, 101*(3), 529–544.

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2009). The importance of prior knowledge when comparing examples: Influences on conceptual and procedural knowledge of equation solving. *Journal of Educational Psychologym, 101*(4), 836–852.

Rogers, C. (1983). *Freedom to learn for the 80's*. Columbus: Charles E. Merill Publishing Company.

Romberg, T. A. (1993). How one comes to know: Models and Theories of the learning of mathematics. In M. Niss (Ed.), *Investigations into assessment in mathematics education* (pp. 97–111). Netherlands: Kluwer Academic Publishers.

Romberg, T. A., Zarinnia, E. A., & Collins, K. F. (1990). A new world view of assessment in mathematics. In G. Kulm (Ed.), *Assessing higher order thinking in mathematics* (pp. 21–38). Washington DC: American Association for the Advancement of Science.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* London: Academic Press, Inc.

Schoenfeld, A. H. (1987). Cognitive science and mathematics education: An overview. In A.H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 1-32). London: Lawrence Erlbaum Associates, Publishers.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334-370). New York: MacMillan.

Siegler, R. S. (1995). How does change occur: a microgenetic study of number conservation. *Cognitive Psychology, 28*(3), 225–273.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Strawhun, B. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal of Mathematical Behavior, 24*, 287–301.

Star, J. R., & Rittle-Johnson, B. (2007). Flexibility in problem solving: The case of equation solving. *Learning and Instruction*, *XX*, 1–15.

Star, J. R., & Seifert, C. (2006). The development of flexibility in equation solving. *Contemporary Educational Psychology, 31*, 280–300.

Stein, M. I. (1974). *Stimulating creativity*. New York: Academic Press.

Stigler, J. W., & Hiebert, J. (1999). The teaching gap: Best ideas from the world's teachers for improving education in the classroom. New York: Free Press.

Torrance, E. P. (1969). *Creativity*. The United States: National Education Association.

Verschaffel, L., Luwel, K., Torbeyns, J., & Dooren, W. V. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, XXIV*(3), 335–359.

Yeap, B.H. (2010). Improving mathematical thinking through assessment. *Journal of Science and Mathematics Education in Southeast Asia, 33*(2), 18